1. A particle (mass $M$, position operator $X$, momentum operator $P$) moves along the $x$ axis under the influence of the Hamilton operator $H = \frac{1}{2M} (P - M\omega X)^2$ where $\omega > 0$ is a constant frequency parameter.

(a) State the Heisenberg equations of motion for $P(t)$ and $X(t)$, and solve them. Then evaluate the commutator $[X(t), X(t_0)]$. [9 marks]

(b) Express $P(t)$, $P(t_0)$, $P(t) - M\omega X(t)$, and $H$ in terms of $X(t)$ and $X(t_0)$. [6 marks]

(c) Find the time transformation function $\langle x, t| x', t_0 \rangle$ by first establishing its derivatives with respect to $x$, $x'$, and $T = t - t_0$. [10 marks]

2. $A$ and $A^\dagger$ are the ladder operators of a harmonic oscillator. A hermitian operator $Z$ is such that $ZA^\dagger = (1 - \lambda)A^\dagger Z$ with $0 < \lambda < 1$, and is normalized to unit trace, $\text{tr} \{Z\} = 1$.

(a) Determine the normally ordered form of $Z$. [15 marks]

(b) Show that $Z$ commutes with $A^\dagger A$. Then express $Z$ as a function of $A^\dagger A$. [10 marks]

3. Orbital angular momentum vector $\vec{L}$ with cartesian components $L_1$, $L_2$, and $L_3$. The system is in an eigenstate of $\vec{L}^2$ with eigenvalue $6\hbar^2$.

(a) What are the possible outcomes when one measures (i) $L_1^2$; (ii) $L_2^2$; (iii) $L_1^2 + L_2^2$? [6 marks]

(b) What are the possible outcomes when one measures $L_1^2 - L_2^2$? [12 marks]

(c) What are the expectation values and the spreads of $L_1$ and $L_2$ in an eigenstate of $L_3$ with eigenvalue $m\hbar$? [7 marks]
4. A harmonic oscillator (mass $M$, natural frequency $\omega$, position operator $X$, momentum operator $P$) is perturbed by a $\delta$-function potential of strength $\propto V$, so that the Hamilton operator is

$$H = H_0 + H_1 \quad \text{with} \quad H_0 = \frac{P^2}{2M} + \frac{1}{2} M \omega^2 X^2 \quad \text{and} \quad H_1 = V \sqrt{\frac{\hbar}{M \omega}} \delta(X),$$

where $\delta(X) = \left( |x\rangle \langle x| \right)_{x=0}$. As usual, we denote the eigenkets of $H_0$ by $|n\rangle$ with $n = 0, 1, 2, \ldots$.

(a) Determine the $\xi = 0$ value of the $n$th Hermite polynomial $H_n(\xi)$ with the aid of the generating function

$$e^{2t\xi - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\xi).$$

Then find $\langle x|n\rangle_{x=0}$ for $n = 0, 1, 2, \ldots$. [10 marks]

(b) Write $E_n(V)$ for the $V$-dependent $n$th eigenvalue of $H$ and determine

$$\frac{\partial E_n}{\partial V} \bigg|_{V=0}.$$

for $n = 0, 1, 2, \ldots$. [10 marks]

(c) Use the large-$m$ approximation $\left( \frac{2m}{m} \right) \approx \frac{4^m}{\sqrt{\pi m}}$ to establish a large-$n$ approximation for $\frac{\partial E_n}{\partial V} \bigg|_{V=0}$. [5 marks]