Problem 1 (20 marks)
A harmonic oscillator (mass $M$, natural frequency $\omega$) is in its ground state. Determine the expectation value of $\frac{1}{2}[X(t_1)X(t_2) + X(t_2)X(t_1)]$ for any two times $t_1$ and $t_2$.

Problem 2 (20 marks)
Orbital angular momentum: If $\vec{L}^2$ has the value $l(l+1)\hbar^2$, with $l = 0, 1, 2, \ldots$, what is the value of
$$\text{tr}\left\{e^{\gamma \vec{L}_3/\hbar}\right\}$$
for real $\gamma$?

Problem 3 (30 marks)
A hydrogenic atom (as usual: electron mass $M$, electron charge $-e$, nuclear charge $Ze$) is exposed to a perturbing potential that is given by
$$H_1 = \frac{V_0}{(r/a_0)^2},$$
where $V_0 > 0$ is the strength of the perturbation and $a_0 = \frac{\hbar^2}{Me^2}$ is the Bohr radius. What is the energy of a bound state with radial quantum number $n_r$ and angular momentum quantum number $l$? [Hint: You can state the exact energy eigenvalues after considering the radial Schrödinger equation.]

Problem 4 (30 marks)
Motion along the $x$ axis; mass $M$, position operator $X$, momentum operator $P$. Use trial wave functions of the form $\psi(x) = \sqrt{\kappa}e^{-\kappa|x|}$, with an adjustable parameter $\kappa > 0$, to establish upper bounds on the ground state energy of the Hamilton operator
$$H = \frac{1}{2M}p^2 - \frac{(\hbar \kappa_0)^2}{M}e^{-\kappa_0|x|},$$
where $\kappa_0 > 0$ specifies the strength and the range of the potential energy. For which value of $\kappa$ do you get the best upper bound? What is its value?