Problem 1 (20 points)
Motion along the $x$ axis. State 1 is specified by its position wave function
\[
\langle x|1\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{a}}e^{-\left(\frac{x^2}{a}\right)} \quad \text{with } a > 0,
\]
and state 2 is specified by its momentum wave function
\[
\langle p|2\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{b}}e^{-\left(\frac{p^2}{b}\right)} \quad \text{with } b > 0.
\]
Calculate the transition probability $|\langle 1|2\rangle|^2$. [Hint: You can run a simple check on your answer, because you know the probability when $2ab = \hbar$.]

Problem 2 (25 points)
Motion along the $x$ axis; position operator $X$, momentum operator $P$.
Consider the Hamilton operator $H = -\Omega(XP + PX)$ with $\Omega > 0$.
(a) Solve the Heisenberg equations of motion, that is: express $X(t)$ and $P(t)$ in terms of $X(t_0)$, $P(t_0)$, and $T = t - t_0$.
(b) Evaluate the commutator $[X(t), P(t_0)]$.
(c) Find first the time transformation function $\langle x, t|p, t_0\rangle$ and then the time transformation function $\langle x, t|x', t_0\rangle$.

Problem 3 (15 points)
Harmonic oscillator; ladder operators $A^{\dagger}$ and $A$; Hamilton operator $H = \hbar\omega A^{\dagger}A$.
At the initial time $t_0$, the statistical operator $\rho(A^{\dagger}, A, t_0)$ is given by the normally ordered exponential
\[
\rho(A^{\dagger}, A, t_0) = e^{-(A^{\dagger} - \alpha^*)(A - \alpha)},
\]
where $\alpha$ is an arbitrary complex number and $\alpha^*$ is its complex conjugate. What is the statistical operator $\rho(A^{\dagger}, A, t)$ at the later time $t = t_0 + T$?

Problem 4 (20 points)
A harmonic oscillator is in the state described by the ket $\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, which is an equal-weight superposition of the Fock states to $n = 0$ and $n = 1$.
(a) What are the expectation values of $A$, $A^{\dagger}$, $A^2$, and $A^{\dagger 2}$?
(b) What are the spreads $\delta X$, $\delta P$ of position operator $X$ and momentum operator $P$? How large is their product $\delta X \delta P$?

Problem 5 (20 points)
(a) Show that
\[
tr \{F\} = \int \frac{drdp}{2\pi\hbar} f\left(-\frac{\ell p}{\hbar}, \frac{x}{\ell}\right)
\]
where $F = f(A^{\dagger}, A)$ is the normally ordered form of operator $F$.
(b) Use this to calculate the trace of $e^{-\lambda A^{\dagger}A}$ with $\lambda > 0$. 