The Heisenberg equations of motion are
\[ \frac{d}{dt} P = -\frac{\partial H}{\partial x}, \quad \frac{d}{dt} x = \frac{\partial H}{\partial P} = \frac{1}{\hbar} P + \gamma X. \]

and are solved by
\[ P(t) = e^{-\gamma t} P(t_0), \]
\[ X(t) = e^{\gamma t} X(t_0) + \frac{e^{\gamma t} - e^{-\gamma t}}{2\gamma \hbar} \cdot P(t_0) \]

with \( T = t - t_0 \).

We have
\[ [X(t_0), P(t_0)] = \left[ e^{\gamma t} X(t_0), P(t_0) \right] = i\hbar e^{\gamma t}. \]

Proceeding from
\[ i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \langle x, t | H \cdot | p, t_0 \rangle \]

we express \( H \) in terms of \( X(t) \) and \( P(t_0) \). This gives
\[ H = \frac{1}{2\hbar} e^{-2\gamma t} P(t_0)^2 + \frac{1}{2} \gamma e^{-\gamma t} (X(t) P(t_0) + P(t_0) X(t)) \]
\[ = \frac{1}{2\hbar} e^{-2\gamma t} P(t_0)^2 + \gamma e^{-\gamma t} X(t) P(t_0) - \frac{i}{\gamma} \frac{\partial}{\partial x}. \]

so that
\[ i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \left( \frac{1}{2\hbar} e^{-2\gamma t} P^2 + \gamma e^{-\gamma t} x p - \frac{i}{\gamma} x \right) \langle x, t | p, t_0 \rangle \]
\[ = \langle x, t | p, t_0 \rangle \frac{\partial}{\partial t} \left( \frac{p^2}{2\hbar} \frac{1-e^{-2\gamma t}}{2\gamma} + xp(1-e^{\gamma t}) - \frac{i}{2\gamma} x t \right). \]
As a consequence, we get
\[
\langle x, t | p, t_0 \rangle = \frac{1}{\sqrt{2 \pi \hbar}} e^{-\frac{x^2}{2\hbar}} e^{\frac{i p x}{\hbar}} e^{-\frac{i p^2}{2m}\frac{1-e^{-2\hbar}}{2\hbar}}
\]

upon incorporating the initial condition
\[
\langle x, t_0 | p, t_0 \rangle = \frac{1}{\sqrt{2 \pi \hbar}} e^{\frac{i p x}{\hbar}}.
\]

(2a) For \( \hat{H} = \hat{H}_0 + \hat{H}_1 \), with \( \hat{H}_0 = \frac{1}{2} \omega \hat{A}^+ \hat{A} \) and \( \hat{H}_1 = \frac{1}{2} \Omega(A^+ \hat{A}^2 + \text{h.c.}) \), the unperturbed states \( |n^{(0)}\rangle \) are the standard Fock states. The relevant matrix elements of \( \hat{H}_1 \) are those between the ground state \( |0^{(0)}\rangle \) and any excited state \( |m^{(0)}\rangle \),
\[
\langle m^{(0)} | \hat{H}_1 | 0^{(0)} \rangle = \frac{1}{2} \langle m^{(0)} | \hat{A}^+ \hat{A}^2 | 0^{(0)} \rangle
\]
\[
= \frac{1}{2} \sqrt{2 \omega^2 \delta m_1 2}.
\]

Accordingly, the first-order correction vanishes,
\[
E_0^{(1)} = \langle 0^{(0)} | \hat{H}_1 | 0^{(0)} \rangle,
\]

and the second-order correction is
\[
E_0^{(2)} = - \sum_{m=1}^{\infty} \frac{1}{E_m^{(0)} - E_0^{(0)}} \left| \langle m^{(0)} | \hat{H}_1 | 0^{(0)} \rangle \right|^2
\]
\[
= \frac{1}{2} \sum_{m=1}^{\infty} \frac{(\delta m_1 2)^2}{m \hbar \omega} = - \frac{\hbar}{\omega} \delta m_1 2.
\]

(2b) We have (page 57-59 of the notes)
\[
\hat{A}^+ + \hat{A}^2 = \frac{\hbar \omega}{\tilde{a}} X^2 - \frac{1}{\hbar \omega} T^2
\]
and
\[
\hbar \omega \hat{A}^+ \hat{A} = \frac{1}{2 \hbar} P^2 + \frac{1}{2} \hbar \omega^2 X^2 - \frac{1}{2} \hbar \omega T^2.
\]
so that

\[ H = \left( \frac{1}{2M} - \frac{\delta^2}{M\omega} \right) p^2 + \left( \frac{1}{2} M \omega^2 + M \delta \omega \right) x^2 - \frac{1}{2} \hbar \omega \]

\[ = \frac{1}{2M} p^2 + \frac{1}{2} M \bar{\omega}^2 x^2 - \frac{1}{2} \hbar \omega \]

which is a harmonic oscillator with mass \( M \) and frequency \( \bar{\omega} \) that are given by

\[ \frac{1}{M} = \left( \frac{1}{M} - \frac{\delta^2}{M\omega} \right)^{-1}, \quad \bar{\omega}^2 = \omega^2 + 2M\delta \omega. \]

Thus,

\[ \bar{\omega} = \sqrt{\omega^2 - 4\delta^2} \]

and the true ground-state energy is

\[ E_0 = \frac{1}{2} \hbar \bar{\omega} - \frac{1}{2} \hbar \omega = \frac{1}{2} \hbar \omega \left( \sqrt{1 - \frac{4\delta^2}{\omega^2}} - 1 \right) \]

\[ = \frac{1}{2} \hbar \omega \left( -\frac{2\delta^2}{\omega^2} + 5 \left( \frac{\delta \omega}{\omega^2} \right)^2 \right) \]

\[ \approx -\frac{5\delta^2}{\omega} \quad \text{to second order in } \delta. \]

This agrees with the second-order result in (2c), as it should.

(2c) Since \( \bar{\omega}^2 = \omega^2 - 4\delta^2 \), reasonable values of \( \delta \) are in the range

\[-\frac{1}{2} \omega < \delta < \frac{1}{2} \omega.\]
(3a) One has
\[
\langle \mathbf{P}^2 \rangle = \hbar^2 \int dx \left| \frac{d}{dx} f(x) \right|^2 = \hbar^2 \int dx ( - \kappa^2 x f(x))^2
\]
\[
= \frac{\hbar^2}{\sqrt{\pi}} \kappa^5 \int dx \, x^2 e^{-\kappa^2 x^2} = \frac{1}{2} \hbar^2 \kappa^2
\]
and
\[
\langle x^4 \rangle = \int dx \, x^4 f(x)^2 = \frac{\kappa}{\sqrt{\pi}} \int dx \, x^4 e^{-\kappa^2 x^2}
\]
\[
= \frac{3}{4 \kappa^4},
\]
after using \( \int dx e^{-\kappa^2 x^2} = \frac{\sqrt{\pi}}{\kappa} \) and \( x^2 e^{-\kappa^2 x^2} \)
\[
= -\frac{3}{2 \kappa^2} e^{-\kappa^2 x^2} \text{ repeatedly.}
\]

(3b) According to the Rayleigh–Ritz variational principle,
\[
E_0 \leq \langle \mathbf{H} \rangle = \frac{\hbar^2 \kappa^2}{4M} + \frac{3\lambda^2}{4K^4}
\]
for all \( \kappa \). The best upper bound of this kind obtains for the \( \kappa \) value that minimizes the right-hand side. We differentiate (explicitly with respect to \( \kappa^2 \)) to determine \( \kappa^2 \) as the solution of
\[
\frac{\hbar^2}{4M} - \frac{3\lambda^2}{2K^6} = 0, \quad \text{Thus} \quad \kappa^2 = \left( \frac{3\lambda^2}{2M} \right)^{1/3}
\]
and
\[
E_0 \leq \left( \frac{3}{4} \right)^{4/3} \left( \frac{\hbar^2 \lambda^2}{M} \right)^{2/3}
\]

(3c) Here we have
\[
\langle \mathbf{P}^2 \rangle = \frac{3}{2} \hbar^2 \kappa^2 \quad \text{and} \quad \langle x^4 \rangle = \frac{15}{4 \kappa^4}
\]
so that
\[ E_1 \leq \frac{3}{4} \frac{t^2 \kappa^2}{M} + \frac{15}{4} \frac{\lambda^2}{k^4} \]

where the right-hand side is minimal for
\[ \kappa^2 = \left(10 \frac{\lambda^2 m}{\hbar^2}\right)^{\frac{1}{3}} \]

and
\[ E_1 \leq \frac{9}{8} (10)^{\frac{1}{3}} \left(\frac{\hbar^2 \lambda}{M}\right)^{\frac{2}{3}} \]

obtains as the best upper bound of this kind on \( E_1 \).

\( (4a) \) Recall that \( (L_1 \pm i L_2) f(L_3) = f(L_3 \pm i\hbar)(L_1 \pm i L_3) \), so that
\[ e^{-i\frac{\hbar}{\lambda} (L_3 \pm i L_2)} e^{i\frac{\hbar}{\lambda} L_1} = e^{i\frac{\hbar}{\lambda} (L_1 \pm i L_2)} \]

and
\[ e^{i\frac{\hbar}{\lambda} L_1} e^{i\frac{\hbar}{\lambda} L_3} = L_1 \cos \varphi + L_2 \sin \varphi, \]
\[ e^{-i\frac{\hbar}{\lambda} L_2} e^{i\frac{\hbar}{\lambda} L_3} = L_2 \cos \varphi - L_1 \sin \varphi \]

follow. Alternatively, we can differentiate with respect to \( \varphi \) and make use of \( [L_1, L_3] = -i\hbar L_2 \) and \( [L_2, L_3] = i\hbar L_1 \).

\( (4b) \) For \( \varphi = \pi \), we have \( L_1 e^{i\frac{\hbar}{\lambda} L_3} = -e^{i\frac{\hbar}{\lambda} L_3} / L_1 \), so that
\[ L_1 e^{i\frac{\hbar}{\lambda} L_3 / 2} |\ell, m_\ell> = e^{i\frac{\hbar}{\lambda} L_3 / 2} |\ell, m_\ell> (-m_\ell), \]

which states that \( e^{i\frac{\hbar}{\lambda} L_3 / 2} |\ell, m_\ell> \)

is an eigenstate of \( L_1 \) with eigenvalue \( -m_\ell \). It must
therefore be equal to \(|\ell, -m, \ell, m\rangle\) up to a phase factor. Accordingly,

\[ |\langle \ell, m, l, m_3 \rangle| = |\langle \ell, -m, l, m_3 \rangle| . \]

It follows by symmetry that, quite generally, the moduli \(|\langle \ell, m, l, m_3 \rangle|\) do not depend on the signs of \(m\), or \(m_3\).

((c)) In view of the symmetry established in (4b), the four probabilities for \(m_1 = \pm 1\) and \(m_3 = \pm 1\) are the same. Also the four probabilities for \(m_1 = 0, m_3 = \pm 1\) and \(m_1 = \pm 1, m_3 = 0\) are identical. To find the \(m_1 = 0, m_3 = 0\) probability, now consider

\[ L_1 |l = 1, m_3 = 1\rangle = \frac{1}{2} (L_1 + iL_2) + \frac{1}{2} (L_1 - iL_2) |l = 1, m_3 = 1\rangle = |l = 1, m_3 = 0\rangle \frac{\delta}{\sqrt{2}} \]

and

\[ L_1 |l = 1, m_3 = -1\rangle = \frac{1}{2} (L_1 + iL_2) + \frac{1}{2} (L_1 - iL_2) |l = 1, m_3 = -1\rangle = |l = 1, m_3 = 0\rangle \frac{\delta}{\sqrt{2}} , \]

or, after taking the difference,

\[ L_1 |l = 1, m_3 = 1\rangle - |l = 1, m_3 = -1\rangle = 0 . \]

This states that \(|l = 1, m_3 = 0\rangle\) is proportional to this difference and, therefore, we find

\[ |\langle l = 1, m_3 = 0 | l = 1, m_3 = 0 \rangle| = 0 . \]

This is the central entry in the table.
\[ \begin{array}{ccc}
  m_3 \\
  l=1 & +1 & 0 & -1 \\
  | & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \text{Table of } |<l,m_1,m_3>|^2 \\
  m_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \text{for } m_1, m_3 = 0, \pm 1. \\
  -1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\end{array} \]

and all other entries are uniquely determined by the symmetry stated above and the normalization of each row and each column to unit sum.

THE END