Solutions


\[
= A \frac{\partial A^4}{\partial A} + \frac{\partial A^4}{\partial A^2} A = 4 AA^3 + 4 A^3 A \\
\]

2) Consider, for instance, the first component of the first statement,

\[
(R \times I + I \times R) = R_2 L_3 - R_3 L_2 + L_2 R_3 - L_3 R_2
\]

\[
= [R_2, L_3] + [L_2, R_3] = i h R_1 + i h R_1 = 2i h R_1,
\]

and likewise for the second and third component. The argument for the second statement is fully analogous.

3a) We recall

\[
(l_1 \pm i l_2) l(m) = l(m+1) \sqrt{(l+m)(l \pm m+1)}
\]

so that, taking half the sum,

\[
l_1 l(m) = l(m+1) + \frac{1}{2} h \sqrt{(l+m)(l+1)}
\]

\[
+ l(m-1) + \frac{1}{2} h \sqrt{(l+m)(l-1)}
\]

For \(l=2\), in particular,

\[
l_1 |2,2\> = |2,1\> \frac{1}{2} h \sqrt{(2+2)(2-2+1)} = 12,1 \frac{1}{2} h,
\]

\[
l_1 |2,-2\> = |2,-1\> \frac{1}{2} h \sqrt{(2+2)(2-2+1)} = 12,-1 \frac{1}{2} h,
\]

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as well as

\[
L_{12,1} = 12,2 > \frac{1}{2} \sqrt{2-1 \cdot (2+1)} + 12,0 > \frac{1}{2} \sqrt{2+1 \cdot (2+1)}
\]

\[
= 12,2 > \frac{1}{2} + 12,0 > \frac{1}{2} \sqrt{3/2},
\]

\[
L_{1,2,-1} = 12,-2 > \frac{1}{2} \sqrt{2-1 \cdot (2+1)} + 12,0 > \frac{1}{2} \sqrt{2+1 \cdot (2+1)}
\]

\[
= 12,-2 > \frac{1}{2} + 12,0 > \frac{1}{2} \sqrt{3/2},
\]

and

\[
L_{1,2,0} = 12,1 > \frac{1}{2} \sqrt{2-0 \cdot (2+1)} + 12,-1 > \frac{1}{2} \sqrt{2+0 \cdot (2+1)}
\]

\[
= (12,1) + (12,-1) \frac{1}{2} \sqrt{3/2}.
\]

(35) We thus have

\[
L_{1} (12,2 \alpha - 12,0 \beta + 12,-2 \alpha)
\]

\[
= (12,1) + (12,-1) \frac{1}{2} (\alpha - \beta \sqrt{3/2}) = 0
\]

so that

\[
\alpha = \sqrt{\frac{3}{2}} \beta
\]

and \(2|\alpha|^2 + |\beta|^2 = 1\) requires \(4|\beta|^2 = 1\). We choose

\[
\beta = \frac{1}{2} \quad \text{and then get} \quad \alpha = \sqrt{\frac{3}{8}}.
\]

(4a) We have \(\frac{d}{dx} y(x) = -k^2 x^4(x)\) and so get

\[
\langle P^2 \rangle = \int_{-\infty}^{\infty} dx \frac{d^2}{dx^2} y(x) = \frac{1}{h^2} k^5 \int_{-\infty}^{\infty} dx x^2 e^{-k^2 x^2}
\]
\[ \frac{h^2 k^5}{2 \pi} \left( -\frac{1}{2k} \frac{\partial}{\partial k} \right) \int_{-\infty}^{\infty} \sin kx \, e^{-k^2 x^2} \, dk = \frac{\pi}{2} \frac{h^2 k^2}{2} \cdot \]

Further,

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, |x|^2 \, dt \]

\[ y = x^2 \]

\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} y e^{-y^2} \, dy = \frac{1}{\sqrt{\pi} \, k^3} \cdot \]

(4b) We have

\[ E_0 \leq \frac{1}{2M} \langle x^2 \rangle + \frac{k^2}{4M} + \frac{\lambda^2}{4M} + \frac{\pi}{2} \frac{h^2 k^2}{2} \cdot \]

for all \( k > 0 \), so that we get the best upper bound of this kind for the \( k \) value that minimizes the right-hand side. We find it by differentiation:

\[ \frac{h^2 k}{2M} - \frac{3 \lambda^2}{\sqrt{\pi} \, k^4} = 0 \quad \text{or} \quad k^5 = \frac{6 \lambda^2 M}{\pi} \]

and obtain

\[ E_0 \leq \left[ \frac{1}{4} \left( \frac{h^2}{\sqrt{\pi}} \right)^{2/5} + \frac{1}{\sqrt{\pi}} \left( \frac{h^2}{\sqrt{\pi}} \right)^{-3/5} \right] \left( \frac{h^2}{M} \right)^{\lambda/5} \]

\[ = 5 \cdot \frac{2}{5} \cdot \frac{8/5}{3} \cdot \frac{1}{\pi} \cdot \frac{k^2}{M} \cdot \frac{3/5}{4} \cdot \frac{1}{5} \]

\[ = \frac{5 \cdot \left( \frac{h^2 \lambda^2}{M^2} \right)^{1/5}}{(6812 \pi)^{1/5}} \cdot \]
(5a) We have

\[ H = \hbar \omega A^+ A - \frac{e^2}{\sqrt{2}} (A^+ A) \frac{l}{\hbar} \]

and the unperturbed states \( |n^{(0)}\rangle \) are the Fock states, the eigenstates of \( A^+ A \). The relevant matrix elements of \( H \) are

\[ \langle m^{(0)} | H | n^{(0)} \rangle \quad \text{for} \quad n = 0, \]

that is

\[ \langle m^{(0)} | (- \frac{e^2}{\sqrt{2}}) (A^+ A) | 0^{(0)} \rangle = - \frac{e^2}{\sqrt{2}} \delta_{m,1} \]

since \( A | 0^{(0)} \rangle = 0 \) and \( A^+ | 0^{(0)} \rangle = | 1^{(0)} \rangle \).

Accordingly, the 1st order change vanishes

\[ E^{(1)} = \langle 0^{(0)} | H | 0^{(0)} \rangle = 0 \]

and the 2nd order change is

\[ E^{(2)} = - \sum_{m \neq 0} \frac{| \langle m^{(0)} | H | 0^{(0)} \rangle |^2}{E^{(0)} - E^{(0)}} = - \frac{(e^2 \sqrt{2})^2}{\hbar \omega} \]

\[ = - \frac{1}{2} \frac{e^2 \omega^2}{\hbar \omega} = - \frac{1}{2} \frac{e^2}{M \omega^2} \]

where the unperturbed energies \( E_m^{(0)} = \hbar \omega m \) are taken into due account. Thus to 2nd order in \( F \) we have

\[ E_0 = E_0^{(0)} + E_0^{(1)} + E_0^{(2)} = - \frac{1}{2} \frac{e^2}{M \omega^2} \]
(56) We complete the square in $H$,

$$H = \frac{1}{2M} \dot{P}^2 + \frac{1}{2} M \omega^2 \left( x - \frac{E}{M \omega^2} \right)^2 - \frac{E^2}{2M \omega^2} - \frac{1}{2} \hbar \omega$$

$$= \overline{H} - \frac{E^2}{2M \omega^2}$$

where

$$\overline{H} = \frac{1}{2M} \dot{P}^2 + \frac{1}{2} M \omega^2 \left( x - \frac{E}{M \omega^2} \right)^2 - \frac{1}{2} \hbar \omega$$

is the Hamilton operator of a harmonic oscillator located at $x = \frac{E}{M \omega^2}$ rather than $x = 0$. This change in location has, of course, no effect on the eigenvalues, so that $\overline{H}$ has the same eigenvalues as $H_0 = \frac{1}{2M} \dot{P}^2 + \frac{1}{2} M \omega^2 x^2 - \frac{1}{2} \hbar \omega$. In particular, the ground state of $H$ has energy 0. Accordingly, the exact ground state energy of $\overline{H}$ is

$$E_0 = -\frac{E^2}{2M \omega^2}$$

which is identical with the 2nd-order perturbation result of (50).